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Hydrodynamic Theory of Electron Transport in a Strong Magnetic Field

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The mode coupling contribution to the transverse transport coefficients of a three-dimensional one-component plasma in a strong external magnetic field is calculated. For very strong fields it is found that the tagged particle diffusion rate, the thermal diffusion rate, and the coefficient of viscosity in the plane orthogonal to the field have a Bohm-like $\sim B^{-1}$ behavior. The mode coupling mechanism responsible for such an effect is always one that involves the finite-frequency upper hybrid modes.

KEY WORDS: Electron gas; mode coupling theory; transport; magnetic field.

1. INTRODUCTION

In this paper the dependence of the transport coefficients of a classical one-component plasma on the strength of an applied uniform magnetic field **B** is studied. Particular attention is given to the regime of strong magnetic field, where the cyclotron frequency of the carriers exceeds the rate of collisions. In this limit the contribution of hydrodynamic fluctuations to the Green–Kubo time correlation function expressions for the transport coefficients is evaluated by using a mode coupling theory. We find that a coupling mechanism that was neglected by previous authors is responsible for an enhancement of the transverse coefficients of self-diffusion, heat con-

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ductivity, and shear viscosity in three dimensions. The magnetic field dependence of our results is general. The coefficients of these contributions are, however, evaluated only for small plasma parameter. The main results of this work have been given elsewhere.[1]

There is considerable experimental and theoretical interest in the transport properties of fluids with charged carriers in a magnetic field. From a theoretical point of view[2-9] the understanding of such processes is incomplete, even in the simplest case of a fluid with only one species of carrier. The classical kinetic theory, based on the Balescu–Guernsey–Lenard (BGL) equation, predicts that in the strong-field regime the transport coefficients in the direction parallel to the field are unaffected, while those describing transport in the plane orthogonal to the field decrease as B^{-2} (see, e.g., Refs. 10 and 11). These predictions are in disagreement with both laboratory[12,13] and computer experiments[4,5,8] in a variety of systems. The transverse transpot rates are usually observed to be much larger than those predicted by the BGL equation and to behave as B^{-1} rather than B^{-2} for strong fields. In plasmas the B^{-1} strong-field behavior of the diffusion coefficients is known as Bohm diffusion (see, e.g., Ref. 14).

The BGL equation only contains the effect of uncorrelated effective two-body collisions and describes the decay of spontaneous thermal fluctuations whose lifetime is small compared to the mean free time between collisions. Due to more complex dynamically correlated collision sequences the fluid can also sustain long-wavelength collective fluctuations or hydrodynamic modes that are long-lived on the kinetic time scale. The importance of the long-lived hydrodynamic fluctuations in determining the transport properties was first recognized for the case of neutral fluids, where they were shown to be responsible for the increase of the transport coefficients near the critical point[15] and for the long-time algebraic decay of the time correlation functions determining the transport coefficients (for areview see Ref. 16). The relevance of the same physical mechanism to the transport properties of charged fluids has also been recognized before,[4,6] but its consequences have not yet been fully exploited. When an external magnetic field is applied, the particles are forced to gyrate in Larmor orbits around the field lines. For strong fields the mean free path of the carriers in the plane orthogonal to the field is effectively reduced to a value of the order of the Larmor radius r_L, with $r_L = (k_B T/m)^{1/2}\omega_B^{-1}$, where $\omega_B = e|B|/mc$ is the cyclotron frequency. As a consequence, the range of applicability of hydrodynamics in the direction transverse to the applied field is greatly extended. The hydrodynamic contribution to the transport coefficients is expected to increase considerably. For a strong enough field it can even dominate the BGL contribution.

The transport properties of a plasma in a strong magnetic field have been studied by several authors.[2-8] Krommes and Oberman[6] have employed a mode coupling theory to evaluate the magnetic field dependence of the transport coefficients of a one-component plasma in two dimensions. Their results are in agreement with the two-dimensional computer simulations of Okuda and Dawson.[4] In their calculation Krommes and Oberman discard *a priori* a possible coupling mechanism involving the finite-frequency modes of the system. These are modes where the part of the dispersion relation that describes propagation does not vanish with the characteristic wavevector of the mode. Such modes appear in Coulomb systems due to both the long range of the potential and, in the case considered here, the presence of the external magnetic field. In two dimensions the finite-frequency modes known as upper hybrid modes indeed play a less important role in determining the transport properties than the purely diffusive modes. In three dimensions they are, however, responsible for the most important truly three-dimensional (i.e., size-independent) contribution to the transverse transport coefficients, as shown below. Previous studies of the three-dimensional case[3] have also neglcted the finite-frequency upper hybrid modes. They have concentrated on a size-dependent, effctively two-dimensional, effect. A more complete comparison with the literature will be given in Section 7.

In the first part of the paper we discuss the magnetohydrodynamic modes of a one-component plasma in three dimensions. They are an essential ingredient for our calculation. The hydrodynamic modes have been derived using a formal projection operator technique.[17]

In contrast with the neutral fluid case, the transport coefficients appearing in the hydrodynamic dispersion relations of the OCP are finite-frequency transport coefficients, as was pointed out by Baus.[18] This means that the correct dispersion relations cannot be obtained from the phenomenological magnetohydrodynamic equations with frequency-independent transport coefficients.[11]

In Section 2 we derive the hydrodynamic equations using a projection operator technique and obtain the Green–Kubo formula for the transport coefficients.

The dispersion relation for the five hydrodynamic modes of the fluid and for the mode of self-diffusion in the limit of strong magnetic field are given and discussed in Sections 3 and 4, respectively. In Section 5 we evaluate the two-mode coupling contributions to the coefficient of self-diffusion, the coefficient of thermal conductivity, and the five kinematic viscosities. For very strong field the two-mode coupling theory no longer applies and the effect of higher order mode coupling effects needs to be taken into account. This is done approximately in Section 6 through a "self-consistent" mode coupling theory.

The relationship of our work to previous studies and the experimental relevance of our results are discussed in Section 7.

2. GENERALIZED HYDRODYNAMICS OF THE OCP IN AN EXTERNAL MAGNETIC FIELD

We consider a system of N charged particles of mass m and charge e, immersed in a uniform neutralizing background of opposite charge and placed in an external uniform magnetic field $\mathbf{B}$. The hydrodynamic equations can be obtained from the Liouville equation by using a formal projection operator technique.[17] This method has the advantage of being formally exact—it is not restricted to small values of the plasma parameter—and of providing us with the Green–Kubo formulas for the transport coefficients. These expressions will be the starting point for evaluating the hydrodynamic mode coupling contribution to the transport coefficients.

The hydrodynamic equations for the system considered are equations for the average mass, momentum, and energy densities, defined as the ensemble averages of the microscopic mass density $\rho(\mathbf{r})$, momentum density $\mathbf{g}(\mathbf{r})$, and energy density $\varepsilon(\mathbf{r})$.

It is convenient here to consider the Fourier transforms of the microscopic densities, defined as

$$a_{\mathbf{k}} = \int_{\Omega} d\mathbf{r} [\exp(-i\mathbf{k}\cdot\mathbf{r})][a(\mathbf{r}) - \langle a(\mathbf{r})\rangle_{\text{eq}}] \tag{2.1}$$

where a denotes any of the densities, the brackets denote the average over an equilibrium grand canonical ensemble, and Ω is the volume of the system. From translational invariance of the equilibrium averages we find

$$\rho_{\mathbf{k}} = \sum_{n=1}^{N} m \exp(-i\mathbf{k}\cdot\mathbf{r}_n) - \Omega\rho\delta_{\mathbf{k},0} \tag{2.2a}$$

$$\mathbf{g}_{\mathbf{k}} = \sum_{n=1}^{N} m\mathbf{v}_n \exp(-i\mathbf{k}\cdot\mathbf{r}_n) \tag{2.2b}$$

$$\varepsilon_{\mathbf{k}} = \sum_{n=1}^{N} \varepsilon_n \exp(-i\mathbf{k}\cdot\mathbf{r}_n) - \Omega\varepsilon\delta_{\mathbf{k},0} \tag{2.2c}$$

where $\delta_{\mathbf{k},0}$ is a Kronecker delta, and $\rho = \langle\rho(\mathbf{r})\rangle_{\text{eq}}$ and $\varepsilon = \langle\varepsilon(\mathbf{r})\rangle_{\text{eq}}$ are the average mass and energy density, with $\rho = mn$ and n the number density. Here $\mathbf{r}_n$ and $\mathbf{v}_n$ denote the position and velocity of the nth particle. The particles are assumed to interact through a pairwise additive central Coulomb potential $V(r_{nm})=e^2/r_{nm}$, with $r_{nm}=|\mathbf{r}_n-\mathbf{r}_m|$; ε_n is the energy of the nth pair, given by[(18,19),4]

$$\varepsilon_n=\frac{1}{2}mv_n^2+\frac{1}{2}\sum_{\substack{m=1\\m\neq n}}^{N}\frac{1}{\Omega}\sum_{\mathbf{k}'\neq 0,\mathbf{k}}v(\mathbf{k},\mathbf{k}')\exp(-i\mathbf{k}'\cdot\mathbf{r}_{nm}) \tag{2.3a}$$

where

$$v(\mathbf{k},\mathbf{k}')=V_{k'}\frac{\mathbf{k}'\cdot(\mathbf{k}'-\mathbf{k})}{|\mathbf{k}-\mathbf{k}'|^2} \tag{2.3b}$$

and V_k is the Fourier transform of the Coulomb potential, $V_k=4\pi e^2/k^2$.

Instead of using the microscopic densities given in Eqs. (2.2), it is covenient to introduce the following linear combinations:

$$a_{1,\mathbf{k}}=\rho_{\mathbf{k}} \tag{2.4a}$$

$$a_{j,\mathbf{k}}=\hat{\mathbf{e}}_j(\hat{\mathbf{k}},\ \hat{\mathbf{b}})\cdot\mathbf{g}_{\mathbf{k}} \qquad \text{for} \quad j=2,3,4 \tag{2.4b}$$

$$a_{5,\mathbf{k}}=T_{\mathbf{k}}=\left.\frac{\partial T}{\partial\rho}\right|_{\varepsilon}\rho_{\mathbf{k}}+\left.\frac{\partial T}{\partial\varepsilon}\right|_{n}\varepsilon_{\mathbf{k}} \tag{2.4c}$$

where $\hat{\mathbf{e}}_j(\hat{\mathbf{k}},\hat{\mathbf{b}})$, for $j=2,3,4$, are three mutually orthogonal unit vectors given by

$$\hat{\mathbf{e}}_1(\hat{\mathbf{k}},\hat{\mathbf{b}})=\hat{\mathbf{b}} \tag{2.5a}$$

$$\hat{\mathbf{e}}_2(\hat{\mathbf{k}},\hat{\mathbf{b}})=(1/k_\perp)[\mathbf{k}-(\hat{\mathbf{b}}\cdot\mathbf{k})\hat{\mathbf{k}}] \tag{2.5b}$$

$$\hat{\mathbf{e}}_3(\hat{\mathbf{k}},\hat{\mathbf{b}})=(1/k_\perp)(\hat{\mathbf{b}}\times\hat{\mathbf{k}}) \tag{2.5c}$$

with $\hat{\mathbf{b}}=\mathbf{B}/|\mathbf{B}|$, $k_\perp^2=k_x^2+k_y^2$, and $\hat{\mathbf{k}}=\mathbf{k}/|\mathbf{k}|$. In the following the $\hat{\mathbf{b}}$ dependence of the $\hat{\mathbf{e}}_j$ will not be indicated, unless necessary for clarity. In Eq. (2.4c) $T_{\mathbf{k}}$ represents the temperature fluctuation.

The time evolution of the microscopic densities is formally given by the solution of the Liouville equation

$$\frac{\partial}{\partial t}a_{\mathbf{k}}(t)=L_B a_{\mathbf{k}}(t) \tag{2.6}$$

where L_B is the Liouville operator for the system.

$$L_B=L+\omega_B\sum_{n=1}^{N}R_b(\mathbf{v}_n) \tag{2.7a}$$

[4] It is possible, starting with the Liouville operator for a two-component system, to show that the effect of the uniform background can be incorporated by subtracting the $\mathbf{k}'=0$ term in the sum in Eq. (2.5c).

Here L is the Liouville operator for the system in the absence of external field, given elsewhere,[22] and

$$R_b(\mathbf{v}_n) = \mathbf{v}_n \times \hat{\mathbf{b}} \cdot \partial/\partial \mathbf{v}_n \tag{2.7b}$$

with $\omega_B = e|B|/mc$ the cyclotron frequency. In the following the field **B** will be chosen to point in the z direction, i.e., $\hat{\mathbf{b}} = \hat{\mathbf{z}}$. An inner product between phase functions is defined as

$$(a_{\mathbf{k}}, b_{-\mathbf{k}}) = \lim_{\substack{N,\Omega \to \infty \\ N/\Omega = n}} \frac{1}{\Omega} \langle a_{\mathbf{k}} b_{-\mathbf{k}} \rangle_{\text{eq}} \tag{2.8a}$$

where we have used translation invariance of the equilibrium averages. The set of five variables $a_{j,\mathbf{k}}$ are approximately mutually orthogonal,[5] but not orthonormal,

$$(a_{1,-\mathbf{k}}, a_{1,\mathbf{k}}) = m\rho S(k) \tag{2.9a}$$

$$(a_{j,-\mathbf{k}}, a_{j,\mathbf{k}}) = \rho k_{\text{B}} T \tag{2.9b}$$

$$(a_{5,-\mathbf{k}}, a_{5,\mathbf{k}}) = (k_{\text{B}} T^2/\rho C_v) + O(k^2) \tag{2.9c}$$

Here $C_v = (1/\rho)(\partial\varepsilon/\partial T)|_\rho$ is the specific heat at constant volume per unit mass, $S(k)$ is the static structure factor, related to the pair correlation function $h(k)$ and to the direct correlation function $C(k)$ by[6]

$$S(k) = 1 + nk(k) = [1 - C(k)]^{-1} \tag{2.10}$$

For a system of neutral particles the small-k limit of $S(k)$ defines the isothermal compressibility $\chi_T = n^{-1}(\partial n/\partial p)|_T$, with p the equilibrium pressure, according to $\lim_{k \to 0} S(k) = \chi_T/\chi_T^0$, where $\chi_T^0 = \beta/n$ is the compressibility of an ideal gas. For an electron gas $S(k) \sim k^2\lambda_{\text{D}}^2$ for small k, where $\lambda_{\text{D}} = (4\pi ne^2\beta)^{-1/2}$ is the Debye length, and the isothermal compressibility is defined by[20]

$$S(k) \xrightarrow[k\lambda_{\text{D}} \ll 1]{} \left(\frac{1}{k^2\lambda_{\text{D}}^2} + \frac{\chi_T^0}{\chi_T}\right)^{-1} \tag{2.11}$$

Here we conclude our introduction on notation and definitions and proceed to discuss the equations of generalized hydrodynamics.

[5] Neglecting terms of $O(k^2)$ in the normalization of the densities does not affect our results. This is because the mode coupling contributions to the transport coefficients that are most important in the limit of strong field only involve the kinetic part of the $a_{j\mathbf{k}}$. For these kinetic parts the orthogonality condition is exact to all orders in k.

[6] For a review of the static properties of the OCP see Baus and Hansen.[20]

The hydrodynamic equations have been obtained by using a standard projection operator technique. The derivation is analogous to that of the equations for the neutral fluid case.[17] The details are not given here. Our explicit calculations are restricted to a low-density OCP, where the coefficients of bulk viscosity vanish.[11] For this reason we neglect the bulk viscosities in our derivation. The average densities $\langle a_{j,\mathbf{k}}(t)\rangle$ are defined as averages of the $a_{j,\mathbf{k}}(t)$ over the initial (nonequilibrium) ensemble of the system. The resulting equations for the Laplace-transformed average densities, defined as

$$\langle \tilde{a}_{j,\mathbf{k}}(z)\rangle = \int_0^\infty dt\, e^{-zt}\langle a_{j,\mathbf{k}}(t)\rangle \tag{2.12}$$

for Re $z > 0$, are given by

$$\sum_{j=1}^{5} [z\delta_{ij} + \omega_B B_{ij} + ik\Omega_{ij}(\mathbf{k}) + k^2 U_{ij}(\mathbf{k}, z)]\langle \tilde{a}_{j,\mathbf{k}}(a)\rangle$$
$$= \langle a_{j,\mathbf{k}}(t=0)\rangle \tag{2.13}$$

The only nonvanishing elements of the constant matrix B_{ij} are

$$B_{34} = -B_{43} = 1 \tag{2.14}$$

The matrices Ω_{ij} and U_{ij} are more conveniently expressed in terms of the microscopic mass current $\mathbf{j}^\rho_\mathbf{k}$, momentum current $j^g_{\alpha\beta,\mathbf{k}}$, and energy current $\mathbf{j}^e_\mathbf{k}$, whose definition has been given elsewhere.[19,22,23] The fluxes associated with $a_{i,\mathbf{k}}$ are linear combinations of these and are defined by

$$\mathbf{j}^1_\mathbf{k} = \mathbf{j}^\rho_\mathbf{k} \tag{2.15a}$$

$$j^i_{\alpha,\mathbf{k}} = \hat{e}_{i\beta}(\hat{\mathbf{k}})\, j^g_{\alpha\beta,\mathbf{k}} \qquad \text{for} \quad i = 2, 3, 4 \tag{2.15b}$$

$$\mathbf{j}^5_\mathbf{k} = \frac{1}{\rho C_v}\left(\mathbf{j}^\varepsilon_\mathbf{k} - \rho C_v \left.\frac{\partial T}{\partial \rho}\right|_\varepsilon \mathbf{j}^\rho_\mathbf{k}\right) \tag{2.15c}$$

where

$$\rho C_v \left.\frac{\partial T}{\partial \rho}\right|_\varepsilon = \frac{\alpha T}{\rho \chi_T} - \frac{h}{\rho}$$

with $h = \varepsilon + p$ the enthalpy per unit volume and $\alpha = -n^{-1}(\partial n/\partial T)|_p$ the coefficient of thermal expansion. The frequency matrix Ω_{ij} can be written as

$$\Omega_{ij}(\mathbf{k}) = (a_{i,-\mathbf{k}}, \hat{\mathbf{k}} \cdot \mathbf{I}^i_\mathbf{k})(a_{j,-\mathbf{k}}, a_{j,\mathbf{k}})^{-1} \tag{2.16}$$

and it has the property

$$\Omega_{ij}(\mathbf{k})(a_{j,-\mathbf{k}}, a_{j,\mathbf{k}}) = -\Omega^*_{ji}(\mathbf{k})(a_{i,-\mathbf{k}}, a_{i,\mathbf{k}}) \tag{2.17}$$

The transport matrix is given by

$$U_{ij}(\mathbf{k}, z) = \hat{k}_\alpha \hat{k}_\beta \left(I^i_{\alpha,-\mathbf{k}}, \frac{1}{z - \hat{L}_B} I^j_{\beta,\mathbf{k}} \right) (a_{j,-\mathbf{k}}, a_{j,\mathbf{k}})^{-1} \tag{2.18}$$

and vanishes for $i=1$ or $j=1$. Here $\hat{L}_B$ is a modified or projected Liouville operator defined elsewhere[15] and $\mathbf{I}^i_{\mathbf{k}}$ are the projected currents,[23] given by $I^i_{\alpha,\mathbf{k}} = \hat{e}_{i\beta}(\hat{\mathbf{k}})\, I^g_{\alpha\beta,\mathbf{k}}$, for $i=2, 3, 4$, with

$$I^g_{\alpha\beta,\mathbf{k}} = j^g_{\alpha\beta,\mathbf{k}} - \delta_{\alpha\beta} \left\{ [m\beta S(k)]^{-1} \rho_{\mathbf{k}} - \frac{(\gamma-1)\,\rho C_v}{\alpha_T} T_{\mathbf{k}} - p\Omega\delta_{\mathbf{k},\mathbf{0}} \right\} \tag{2.19a}$$

and

$$I^5_{\alpha,\mathbf{k}} = j^5_{\alpha,\mathbf{k}} - (h/\rho^2 C_v)\, j^1_{\alpha,\mathbf{k}} \tag{2.19b}$$

To make connection with the usual transport coefficients, U_{ij} can be expressed in terms of generalized transport tensors that satisfy certain symmetry properties (Onsager's relations). The general expressions are not given here.

The usual hydrodynamic equations describe the time evolution of the average densities for small values of k and of the frequency $\omega = iz$. Similarly, here we replace the transport matrix U_{ij} with its $k \to 0$ limit. In contrast with neutral fluids, we need to keep the frequency dependence of the transport coefficients because we expect that some of the hydrodynamic modes of the OCP will be finite-frequency modes, i.e., modes whose propagating part does not vanish with k. This point is not merely a technical detail, because the density and, in our case, magnetic field dependence of the finite-frequency transport coefficients can be qualitatively different from that of their zero-frequency form. A precise definition of the length and time scales relevant to the problem considered here will be given below. The five magnetohydrodynamic equations are explicitly given by

$$z\langle \tilde{n}_{\mathbf{k}}(z)\rangle + ik_z n\langle \tilde{a}_{2,\mathbf{k}}(z)\rangle + ik_\perp n\langle \tilde{a}_{3,\mathbf{k}}(z)\rangle = \langle n_{\mathbf{k}}\rangle \tag{2.20a}$$

$$\begin{aligned}
&\left[z + \frac{4}{3}\nu_0(z)k_z^2 + \nu_2(z)k_\perp^2 \right] \langle \tilde{a}_{2,\mathbf{k}}(z)\rangle - ik_z \frac{\alpha}{\rho\chi^T} \langle \tilde{T}_{\mathbf{k}}(z)\rangle \\
&\quad + \frac{ik_z}{n}\left(\frac{\omega_p^2}{k^2} + \frac{1}{\rho\chi_T} \right) \langle \tilde{n}_{\mathbf{k}}(z)\rangle - \nu_4(z)\, k_z k_\perp \langle \tilde{a}_{4,\mathbf{k}}(z)\rangle \\
&\quad + \left[\nu_2(z) - \frac{2}{3}\nu_0(z) \right] k_z k_\perp \langle \tilde{a}_{3,\mathbf{k}}(z)\rangle = \langle a_{2,\mathbf{k}}\rangle
\end{aligned} \tag{2.20b}$$

$$\left\{z+\nu_2(z)k_z^2+\left[\nu_1(z)+\frac{1}{3}\nu_0(z)\right]k_\perp^2\right\}\langle\tilde{a}_{3,\mathbf{k}}(z)\rangle+ik_\perp\frac{\alpha}{\rho\chi_T}\langle\tilde{T}_\mathbf{k}(z)\rangle$$
$$+\frac{ik_\perp}{n}\left(\frac{\omega_p^2}{k^2}+\frac{1}{\rho\chi_T}\right)\langle\tilde{n}_\mathbf{k}(z)\rangle+[\omega_B-\nu_3(z)k_\perp^2-\nu_4(z)k_z^2]\langle\tilde{a}_{4,\mathbf{k}}(z)\rangle$$
$$+\left[\nu_2(z)-\frac{2}{3}\nu_0(z)\right]k_zk_\perp\langle\tilde{a}_{2,\mathbf{k}}(z)\rangle=\langle a_{3,\mathbf{k}}\rangle \qquad (2.20c)$$

$$[z+\nu_2(z)k_z^2+\nu_1(z)k_\perp^2]\langle\tilde{a}_{4,\mathbf{k}}(z)\rangle+\nu_4(z)\,k_zk_\perp\langle\tilde{a}_{2,\mathbf{k}}(z)\rangle$$
$$-[\omega_B-\nu_4(z)k_z^2-\nu_3(z)k_\perp^2]\langle\tilde{a}_{3,\mathbf{k}}(z)\rangle=\langle a_{4,\mathbf{k}}\rangle \qquad (2.20d)$$

$$[z+D_\parallel^T(z)k_z^2+D_\perp^T(z)k_\perp^2]\langle\tilde{T}_\mathbf{k}(z)\rangle+ik_z\frac{\gamma-1}{\alpha}\langle\tilde{a}_{2,\mathbf{k}}(z)\rangle$$
$$+ik_\perp\frac{\gamma-1}{\alpha}\langle\tilde{a}_{3,\mathbf{k}}(z)\rangle=\langle T_\mathbf{k}\rangle \qquad (2.20e)$$

where use has been made of Eq. (2.14) for the small-k form of the static structure factor. The five kinematic viscosities in Eqs. (2.44) are given by the following Green–Kubo formulas:

$$\nu_0(z;\mathbf{B})=\frac{3\beta}{4\rho}\int_0^\infty dt\,e^{-zt}\int\lim_{\mathbf{k}\to 0}(I^g_{zz,-\mathbf{k}}(t),I^g_{zz,\mathbf{k}}) \qquad (2.21a)$$

$$\nu_1(z;\mathbf{B})=\frac{\beta}{\rho}\int_0^\infty dt\,e^{-zt}\int\lim_{\mathbf{k}\to 0}(I^g_{xy,-\mathbf{k}}(t),I^g_{xy,\mathbf{k}}) \qquad (2.21b)$$

$$\nu_2(z;\mathbf{B})=\frac{\beta}{\rho}\int_0^\infty dt\,e^{-zt}\int\lim_{\mathbf{k}\to 0}(I^g_{xz,-\mathbf{k}}(t),I^g_{xz,\mathbf{k}}) \qquad (2.21c)$$

$$\nu_3(z;\mathbf{B})=\frac{\beta}{\rho}\int_0^\infty dt\,e^{-zt}\int\lim_{\mathbf{k}\to 0}(I^g_{xx,\mathbf{k}}(t),I^g_{yz,\mathbf{k}})$$
$$=-\frac{\beta}{\rho}\int_0^\infty dt\,e^{-zt}\int\lim_{\mathbf{k}\to 0}(I^g_{yx,\mathbf{k}}(t),I^g_{xx,\mathbf{k}}) \qquad (2.21d)$$

$$\nu_4(z;\mathbf{B})=\frac{\beta}{\rho}\int_0^\infty dt\,e^{-zt}\int\lim_{\mathbf{k}\to 0}(I^g_{xz,\mathbf{k}}(t),I^g_{yz,\mathbf{k}})$$
$$=-\frac{\beta}{\rho}\int_0^\infty dt\,e^{-zt}\int\lim_{\mathbf{k}\to 0}(I^g_{yz,\mathbf{k}}(t),I^g_{xz,\mathbf{k}}) \qquad (2.21e)$$

The two coefficients of thermal diffusion are given by

$$D_{\|}^{T}(z)=\frac{1}{\rho C_v k_B T^2}\int_0^{\infty} dt\, e^{-zt}\lim_{\mathbf{k}\to 0}(I_{z,-\mathbf{k}}^{5}(t), I_{z,\mathbf{k}}^{5}) \tag{2.22a}$$

$$D_{\perp}^{T}(z)=\frac{1}{\rho C_v k_B T^2}\int_0^{\infty} dt\, d^{-zt}\lim_{\mathbf{k}\to 0}(I_{x,-\mathbf{k}}^{5}(t), I_{x,\mathbf{k}}^{5}) \tag{2.22b}$$

The viscosities ν_3 and ν_4 are not true transport coefficients, because they are nonzero even in the absence of collisions. Therefore, they do not represent dissipation, but originate from the presence of the magnetic field.[11]

3. HYDRODYNAMIC MODES

In this section we discuss the solution of the set of magnetohydrodynamic equations (2.20). It is convenient to rewrite them in matrix form as

$$[z\delta_{ij}+\mathscr{L}_{ij}(\mathbf{k}, z)]\langle \tilde{a}_{i,\mathbf{k}}(z)\rangle = \langle a_{i,\mathbf{k}}\rangle \tag{3.1}$$

where summation over repeated indices is used. The explicit form of $\mathscr{L}_{ij}$ is obtained by comparing Eqs. (3.1) and (2.20). Its frequency dependence is due to the frequency dependence of the transport coefficients.

We express the solution of the set of equations (3.1) in terms of the eigenvalues and eigenfunctions of the matrix $\mathscr{L}_{ij}$. The right eigenvalue problem is defined by

$$\mathscr{L}_{ij}(\mathbf{k}, -z_{\mu})\,|\theta_{\mu j}^{R}(\mathbf{k})\rangle = z_{\mu}\,|\theta_{\mu j}^{R}(\mathbf{k})\rangle \tag{3.2a}$$

where z_{μ} and $\theta_{\mu}^{\mathrm{R}}$ are the eigenvalues and right eigenfunctions of $\mathscr{L}_{ij}$, respectively. Here and below $\mu, \nu,\ldots$ label the hydrodynamic modes. Since $\mathscr{L}_{ij}$ is not self-adjoint, one also needs to solve the corresponding left eigenvalue problem, with eigenfunctions $\theta_{\mu}^{\mathrm{L}}$, defined by

$$\langle \theta_{\mu i}^{\mathrm{L}}(\mathbf{k})|\,\mathscr{L}_{ij}(\mathbf{k}, -z_{\mu}) = \langle \theta_{\mu j}^{\mathrm{L}}(\mathbf{k})|\,z_{\mu} \tag{3.2b}$$

Right and left eigenfunctions are normalized according to

$$\sum_{j=1}^{5}\langle \theta_{\mu j}^{\mathrm{L}}(k)\,|\,\theta_{\nu j}^{\mathrm{R}}(k)\rangle = \delta_{\mu\nu} \tag{3.2c}$$

We are interested here in determining the long-time behavior of the average densities $\langle a_{j,\mathbf{k}}(t)\rangle$. To do this we investigate the behavior of $\mathscr{L}_{ij}(\mathbf{k}, z)$ around $\mathrm{Re}\, z=0$ for small values of k. We look for those poles of the inverse of the operator on the right-hand side of Eq. (3.1) that are in the half-plane $\mathrm{Re}\, z < 0$ closest to the imaginary axis. These are the hydrodynamic poles or modes and they determine the long-time behavior of $\langle a_{j,\mathbf{k}}(t) \rangle$ if $\mathscr{L}_{ij}(\mathbf{k}, z)$ is a regular function of z around $\mathrm{Re}\, z = 0$.[7] They are also defined as those roots of the determinant of the hydrodynamic matrix

$$\det | -z_\mu \delta_{ij} + \mathscr{L}_{ij}(\mathbf{k}, -z_\mu)| = 0 \tag{3.2d}$$

whose real part vanishes as $k \to 0$.

The nonlinear eigenvalue problem defined in Eqs. (3.2) can be solved perturbatively, treating k as a small parameter. In the hydrodynamic regime one is interested in phenomena that occur over wavelengths large compared to the mean free path l between particle collisions. This defines the characteristic length scale of the problem and identifies kl as the dimensionless small parameter.

For a weakly interacting electron gas in the absence of external fields the mean free path is $l = v_0/v_c$, where $v_0 = (m\beta)^{-1/2}$ is the thermal velocity and v_c the collision frequency associated with the linearized Landau operator, $v_c = \omega_p \varepsilon_p \ln(\varepsilon_p^{-1})$, with ε_p the plasma parameter, $\varepsilon_p = (4\pi n \lambda_{\mathrm{D}}^3)^{-1}$, and ω_p the plasma frequency, $\omega_p = v_0/\lambda_{\mathrm{D}}$. Equations (3.2), with $\mathbf{B} = 0$, can be solved perturbatively and it can be shown that eigenvalues and eigenfunctions have well-defined expansions as power series in kl. The resulting hydrodynamic description applies for $k \ll l^{-1}$.

The presence of the external magnetic field introduces several complications. The field introduces anisotropy in the problem and we expect that the properties in the direction of the field will be different from those in the plane orthogonal to the field. The single-particle dynamics along the field is unaffected by the field itself. In this direction the man free path l is the same as when $\mathbf{B} = 0$ and hydroynamics is a good picture only if collisions are frequent.

In the plane orthogonal to $\mathbf{B}$ the particles are forced to gyrate in Larmor orbits of radius $r_{\mathrm{L}} = v_0/\omega_{\mathrm{B}}$ around the field lines when collisions are absent. The field prevents free streaming, just as collisions do in the zero-field situation. The mean distance traveled between collisions in the plane is of the order of the Larmor radius r_{L}. Since $r_{\mathrm{L}}/l = v_c/\omega_B$, a strong field (defined by $v_c/\omega_B \ll 1$) can reduce the mean free path in the transverse direction. This considerably extends the range of applicability of hydrodynamics in the xy plane. These qualitative considerations suggest that the appropriate small parameters to introduce when solving Eqs. (3.2)

[7] The effect of the singularities and branch points of $\mathscr{L}_{ij}(\mathbf{k}, z)$ is discussed, for example, in Ref. 17. They do not contribute to the leading long-time behavior of $\langle a_{j,\mathbf{k}}(t) \rangle$.

perturbatively are $k_z l$ and $k_\perp r_L$. This statement is made more precise as follows. When solving Eqs. (3.2) we assume that each eigenvalue

$$z_\mu = z_\mu(k_z/k^0_{z\mu}, k_\perp/k^0_{\perp\mu}) \tag{3.3}$$

admits a well-defined expansion in powers of $k_z/k^0_{z\mu}$ and $k_\perp/k^0_{\perp\mu}$. The appropriate scaling parameters $k^0_{z\mu}$ and $k^0_{\perp\mu}$ can be different for different modes. They are the maximum values of k_z and $k_\perp$ up to which each approximate dispersion relation can be consistently used. They will be referred to as the "cutoffs" of the hydrodynamic modes. The cutoffs suggested by our previous physical considerations provide an upper limit to the value of the cutoffs for each mode.

In Appendix A the results for the five hydrodynamic eigenvalues up to order k^2 and the right and left eigenfunctions to lowest nonvanishing order in k are given. Here we will give only the eigenvalues in the limit $\omega_p/\omega_B \ll 1$. This is the regime of interest when evaluating the mode coupling corrections to the transport coefficients.

There are five modes: four propagating finite-frequency modes and one purely diffusive mode. They are as follows:

1. Two high-frequency modes, known in the Vlasov limit as the first Bernstein modes, or the upper hybrid modes. The dispersion relation is given for $\omega_p/\omega_B \ll 1$ by

$$\begin{aligned} z_{h\sigma}(\mathbf{k}) = i\sigma\omega_B &\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega_B^2}\hat{k}_\perp^2 + \frac{\gamma}{2\rho\chi_T}\frac{k_\perp^2}{\omega_B^2}\right) \\ &+ k_z^2 v_{\|}(i\sigma\omega_B)\left(1 + \frac{1}{2}\hat{k}_\perp^2\frac{\omega_p^2}{\omega_B^2}\right) + k_\perp^2 v_\perp(i\sigma\omega_B) + O(k^3) \end{aligned} \tag{3.4a}$$

with $\sigma = \pm 1$. When $\mathbf{B} = 0$, these modes reduce to the plasma modes.

2. Two finite-frequency modes, known in the Vlasov limit as the propagating plasma modes, with frequency, for $\omega_p/\omega_B \ll 1$,

$$\begin{aligned} z_{v\sigma}(\mathbf{k}) = i\sigma\omega_p|k_z| &\left(1 + \frac{\gamma}{2\rho\chi_T}\frac{k^2}{\omega_B^2} - \frac{1}{2}\hat{k}_\perp^2\frac{\omega_p^2}{\omega_B^2} - \frac{3\gamma}{4\rho\chi_T}\frac{k_\perp^2}{\omega_B^2}\right) \\ &+ k_z^2 v'_{\|}(i\sigma\omega_p|\hat{k}_z|)\left(1 + \hat{k}_\perp^2\frac{\omega_p^2}{\omega_B^2}\right) + k_\perp^2 v'_\perp(i\sigma\omega_p|\hat{k}_z|) + O(k^3) \end{aligned} \tag{3.4b}$$

When $\mathbf{B} = 0$, these reduce to the shear modes.

3. One diffusive heat mode, with dispersion relation

$$z_H(\mathbf{k}) = D^T_{\|}k_z^2 + D^T_\perp k_\perp^2 + O(k^4) \tag{3.4c}$$

Here $\hat{k}_\perp = k_\perp/k$ and $\hat{k}_z = k_z/k$. In Eqs. (3.4a) and (3.4b) terms of $O(\omega_p^3/\omega_B^3)$ have been neglected.

The viscosities $\nu_{\|}$, $\nu_\perp$, $\nu'_{\|}$, and $\nu'_\perp$ are linear combinations of the five kinematic viscosities ν_j for $j=0, 1,..., 4$ given in Eqs. (2.2). They are defined as follows:

$$\nu_{\|}(i\sigma\omega_B) = \nu_2(i\sigma\omega_B) - i\sigma\nu_4(i\sigma\omega_B) \tag{3.5a}$$

$$\nu_\perp(i\sigma\omega_B) = \nu_1(i\sigma\omega_B) + \frac{1}{6}\nu_0(i\sigma\omega_B) - i\sigma\nu_3(i\sigma\omega_B) \tag{3.5b}$$

and

$$\nu'_{\|}(i\sigma\omega_p|\hat{k}_z|) = \frac{2}{3}\nu_0(i\sigma\omega_p|\hat{k}_z|) \tag{3.6a}$$

$$\nu'_\perp(i\sigma\omega_p|\hat{k}_z|) = \frac{1}{2}\nu_2(i\sigma\omega_p|\hat{k}_z|) + i\sigma\frac{\omega_p}{\omega_B}|\hat{k}_z|\,\nu_4(i\sigma\omega_p|\hat{k}_z|) \tag{3.6b}$$

As expected, they are finite-frequency complex transport coefficients. The thermal conductivities $D_{\|}^T$ and $D_\perp^T$ are the familiar zero-frequency transport coefficients, defined by Eqs. (2.22) with $z=0$.

The viscosities and heat conductivities are evaluated in Appendix B to lowest order in the plasma parameter. The time evolution of the currents in this limit can be approximately described by the Landau kinetic equation. One finds that $\nu_{\|}$, $\nu'_{\|}$, and $D_{\|}^T$ are $O(\omega_B^0)$; $\mathrm{Re}\,\nu_\perp$, $\mathrm{Re}\,\nu'_\perp$, and $D_\perp^T$ are $O(\omega_B^{-2})$; and $\mathrm{Im}\,\nu_\perp$ as well as $\mathrm{Im}\,\nu'_\perp$ are $O(\omega_B^{-1})$. The right and left eigenfunctions associated with the five modes are given in Appendix A. To leading order in ω_p/ω_B, the upper hybrid modes represent velocity fluctuations in the xy plane, while the propagating plasma modes are associated with density fluctuations and fluctuations of velocity in the z direction. The heat mode represents temperature and density fluctuations.

To complete the discussion of the hydrodynamic modes, we need to estimate their cutoffs. To estimate properly the cutoffs for the heat mode and the propagating plasma modes it is sufficient to evaluate the corresponding dispersion relations up to order k^4 and identify the cutoffs on the basis of their definition above. One finds $k_{zH}^0 = k_{zp}^0 \sim l^{-1}$ and $k_{\perp H}^0 \sim r_L^{-1}$, but $k_{\perp p}^0 \sim l^{-1}$. For the upper hybrid modes the situation is more complicated. In replacing the transport matrix in Eq. (2.18) with its $k \to 0$ limit we neglected the free streaming part of the propagator (of order kv_0) compared to the collision part (of order ν_c, for small ε_p) and to the magnetic field part, of order ω_B. When evaluating the transport coefficients relevant to the upper hybrid modes (to be evaluated at the frequency $z = i\sigma\omega_B$) a resonance or cancellation may occur in the propagator between the frequency and the magnetic field part of the Liouville operator. This is seen, for example, in Appendix B in evaluating the transport coefficients for small ε_p. As a consequence, the k dependence of the transport coefficients can only be safely neglected for $kv_0 \ll v_c$. To verify if the upper hybrid mode could be extended up to $k_\perp \sim r_L^{-1}$ we evaluated the dispersion relation for the upper hybrid mode in a simplified model ($k_z = 0$) by keeping the k dependence of the transport coefficients. We find that the mode in question can indeed be extended up to $k^0_{\perp h} \sim r_L^{-1}$. Furthermore, its dispersion relation is well represented by Eq. (3.4a) up to such a value of $k_\perp$. The longitudinal cutoff is again $k^0_{zh} \sim l^{-1}$.

4. SELF-DIFFUSION

To complete the discussion of the transport properties of an electron gas in an external magnetic field, we consider the process of self-diffusion. This is the simplest of all transport phenomena in a fluid. The relevant microscopic density is the probability density $X_\mathbf{k}$ of a tagged particle in the fluid (denoted here as particle 1), given by

$$C_\mathbf{k} = \exp(-i\mathbf{k}\cdot\mathbf{r}_1) - \delta_{\mathbf{k},0} \tag{4.1}$$

whose time evolution is again governed by the Liouville operator L_B. Proceeding as in Section 2, an equation for the Laplace transform of the average density $\langle C_\mathbf{k}(z)\rangle$ is obtained, in the form

$$[z + k_\alpha k_\beta \mathscr{D}_{\alpha\beta}(\mathbf{k}, z; \mathbf{B})]\langle \tilde{C}_\mathbf{k}(z)\rangle \simeq \langle C_\mathbf{k}\rangle \tag{4.2}$$

where again initial correction terms have been neglected and $\mathscr{D}_{\alpha\beta}$ is a generalized self-diffusion tensor, given by

$$\mathscr{D}_{\alpha\beta}(\mathbf{k}, z; \mathbf{B}) = \left(j^s_{\alpha,-\mathbf{k}}, \frac{1}{z - \hat{L}_B^{(1)}} j^s_{\beta,\mathbf{k}} \right) \tag{4.3}$$

with $j^s_{\alpha,\mathbf{k}} = v_{1\alpha}\exp(-i\mathbf{k}\cdot\mathbf{r}_1)$ the tagged particle current. The long-time behavior of $\langle C_\mathbf{k}(t)\rangle$ is again determined by the hydrodynamic poles of the inverse of the operator on the left-handside of Eq. (4.2). These are found by solving, for small k, the eigenvalue problem

$$k_\alpha k_\beta \mathscr{D}^{(0)}_{\alpha\beta}(-z_s; \mathbf{B})|\theta^R_s(\mathbf{k})\rangle = z_s|\theta^R_s(\mathbf{k})\rangle \tag{4.4}$$

together with the corresponding left eigenvalue problem, where

$$\mathscr{D}^{(0)}_{\alpha\beta}(z; \mathbf{B}) = \lim_{\mathbf{k}\to 0} \mathscr{D}_{\alpha\beta}(\mathbf{k}, z; \mathbf{B}) = D_\perp(z)\delta_{\alpha\beta} + [D_\parallel(z) - D_\perp(z)]\,\hat{b}_\alpha\hat{b}_\beta \tag{4.5a}$$

with

$$D_{\|}(z)=\int_0^{\infty} dt\, e^{-zt}\langle v_{1z}(t)\, v_{1z}\rangle_{\text{eq}} \tag{4.5b}$$

$$D_{\perp}(z)=\int_0^{\infty} dt\, e^{-zt}\langle v_{1x}(t) v_{1x}\rangle_{\text{eq}} \tag{4.5c}$$

There is only one purely diffusive mode, the mode of self-diffusion, with eigenvalue

$$z_s(k)=D_{\|}k_z^2+D_{\perp}k_{\perp}^2+O(k^4) \tag{4.6a}$$

and right and left eigenfunctions

$$\theta_s^{\text{R}}(\mathbf{k})=\theta_s^{\text{L}}(\mathbf{k})=1+O(k) \tag{4.6b}$$

In Eq. (4.6a), $D_{\|}$ and $D_{\perp}$ are the coefficients of self-diffusion along the field and in the plane orthogonal to the field, respectively, defined by Eq. (4.5) with $z=0$. They can be evaluated to lowest order in the plasma parameter as described in Appendix B. For $v_c/\omega_B \ll 1$, one obtains

$$D_{\|}^{(0)}=\frac{3\sqrt{\pi}}{m\beta v_c}\sim O(\omega_B^0) \tag{4.7a}$$

$$D_{\perp}^{(0)}=\frac{1}{3\sqrt{\pi}}r_{\text{L}}^2 v_c\sim O(\omega_B^{-2}) \tag{4.7b}$$

The cutoffs for the mode of self-diffusion are $k_{zs}^0 \sim l^{-1}$ and $k_{\perp s}^0 \sim r_{\text{L}}^{-1}$.

5. HYDRODYNAMIC MODE COUPLING CONTRIBUTION TO THE TRANSPORT COEFFICIENTS

The Green–Kubo expressions for the transport coefficients are given by Eqs. (2.21), (2.22), and (4.5). The objective of this section is to evaluate approximately these expressions in the region of strong ($v_c/\omega_B \ll 1$) and very strong ($\omega_p/\omega_B \ll 1$) magnetic field.

Denoting any of the frequency-dependent transport coefficients by $\lambda(z)$, one finds for the corresponding Green–Kubo formula the form

$$\lambda(z)=\int_0^{\infty} dt\, e^{-zt}\lim_{\mathbf{k}\to 0} C_{\alpha\beta}^{\lambda}(\mathbf{k}, t) \tag{5.1a}$$

with

$$C_{\alpha\beta}^{\lambda}(\mathbf{k}, t)=(I_{\alpha,-\mathbf{k}}^{\lambda}(t), I_{\beta,\mathbf{k}}^{\lambda}) \tag{5.1b}$$

Here $I^{\lambda}_{\alpha,\mathbf{k}}$ denotes the αth Cartesian component (for the viscosities the flux is a second-order tensor) of the projected flux associated with λ.

For very short times, i.e., $t \ll \nu_c^{-1}$, the decay of this correlation function is described by the solution to the linearized Balescu–Guernsey–Lenard equation. The corresponding contributions to the transport coefficients, denoted by $\lambda^0(z)$, are evaluated in Appendix B, using the Landau kinetic equation. The transport coefficients in the direction of the field are found to be independent of the magnetic field, while those describing transport in the plane orthogonal to the field behave as ω_B^{-2}.

For longer times, $t \gg \nu_c^{-1}$, following Kadanoff and Swift,[15] we argue that the decay of the projected current correlation functions of Eq. (5.1b) is governed by the long-lived collective excitations in the fluid known as hydrodynamic modes. One then introduces a representation of the time evaluation operator in Eq. (5.1b) on the basis of successively higher products of densities $a_{i,\mathbf{k}}$, whose time decay is assumed to be governed by the hydrodynamic equations. To lowest nonvanishing order one obtains

$$C^{\lambda}_{\alpha\beta}(\mathbf{k}, t) \simeq \frac{1}{2\Omega} \sum_{i,j=1}^{5} \frac{1}{\Omega} \sum_{\mathbf{q}}{}' \frac{\langle I^{\lambda}_{\alpha,\mathbf{k}} a_{i,-\mathbf{q}} a_{j,\mathbf{q}-\mathbf{k}} \rangle_{\text{eq}}}{\langle a_{i,-\mathbf{q}}\, a_{i,\mathbf{q}} \rangle_{\text{eq}}} \times \frac{\langle a_{i,\mathbf{q}}(t)\, a_{j,\mathbf{k}-\mathbf{q}}(t)\, I^{\lambda}_{\beta,-\mathbf{k}} \rangle_{\text{eq}}}{\langle a_{j,\mathbf{q}-\mathbf{k}} a_{j,\mathbf{k}-\mathbf{q}} \rangle_{\text{eq}}} \tag{5.2}$$

where we have used translational invariance of the equilibrium averages. Equation (5.2) is known as the two (hydrodynamic)-mode coupling approximation for the long-time behavior of $C^{\lambda}_{\alpha\beta}(t)$. A microscopic derivation of Eq. (5.2) has been given for neutral fluids of low density by using a systematic kinetic theory.[8] Recently Marchetti and Kirkpatrick[24] used analogous techniques to establish the same results for charged fluids in the absence of external fields. It is convenient to rewrite Eq. (5.2) in terms of hydrodynamic modes, defined as those linear combinations of the microscopic densities whose time decay is governed by a single exponential with relaxation time $[z_\mu(k)]^{-1}$, with $\mu, \nu, ... = h\sigma, \nu\sigma, H$. The hydrodynamic modes are

$$A_{\mu,\mathbf{k}}(t) = \sum_{j=1}^{5} \theta^{L}_{\mu j}(\mathbf{k})\, a_{j\mathbf{k}}(t) \tag{5.3a}$$

with

$$A_{\mu,\mathbf{k}}(t) = \{\exp[-z_\mu(\mathbf{k})t]\}\, A_{\mu\mathbf{k}} \tag{5.3b}$$

[8] The effect of the singularities and branch points of $\mathscr{L}_{ij}(\mathbf{k}, z)$ is discussed, for example, in Ref. 15. They do not contribute to the leading long-time behavior of $\langle a_{j,\mathbf{k}}(t) \rangle$.

Using the orthogonality properties of right and left eigenfunctions, one easily inverts Eq. (5.3a). Substituting the result into Eq. (5.2), we obtain

$$C^{\lambda}_{\alpha\beta}(\mathbf{k}, t) \simeq \frac{1}{2} \sum_{\mu,\nu = h\sigma,\nu\sigma,H} \frac{1}{\Omega} \sum_{\mathbf{k}}{}' \exp\{-[z_\mu(\mathbf{q}) + z_\nu(\mathbf{k}-\mathbf{q})]t\}$$
$$\times S^{+\mu\nu}_{\lambda\alpha}(\mathbf{q}, \mathbf{k}-\mathbf{q})\, S^{\mu\nu}_{\lambda\beta}(\mathbf{q}, \mathbf{k}-\mathbf{q}) \tag{5.4a}$$

with

$$S^{+\mu\nu}_{\lambda\alpha}(\mathbf{q}, \mathbf{k}-\mathbf{q}) = (A^+_{\mu,-\mathbf{q}} A^+_{\nu,\mathbf{q}-\mathbf{k}}, I^{\lambda}_{\alpha\mathbf{k}}) \tag{5.4b}$$

and

$$S^{\mu\nu}_{\lambda\beta}(\mathbf{q}, \mathbf{k}-\mathbf{q}) = (A_{\mu,\mathbf{q}} A_{\nu,\mathbf{k}-\mathbf{q}}, I^{\lambda}_{\beta,-\mathbf{k}}) \tag{5.4c}$$

where we have introduced the adjoint hydrodynamic modes $A^+_{\mu\mathbf{q}}$, defined as

$$A^+_{\mu,-\mathbf{q}} = \sum_{j=1}^{5} \theta^{\mathrm{R}}_{\mu j}(\mathbf{q}) \frac{a_{j,-\mathbf{q}}}{(a_{j,-\mathbf{q}}, a_{j,\mathbf{q}})} \tag{5.4d}$$

The quantities $S^{+\mu\nu}_{\lambda\alpha}$ and $S^{\mu\nu}_{\lambda\beta}$ will be referred to as the mode coupling amplitudes. In the following we will need the explicit expression of the modes to leading order in ω_p/ω_B. This is easily obtained from Appendix A. or the upper hybrid modes one has

$$A_{h\sigma,\mathbf{q}} = \frac{1}{\sqrt{2}} [-\sigma \hat{\mathbf{e}}_3(\hat{\mathbf{q}}) \cdot \mathbf{g}_{\mathbf{q}} - i\hat{\mathbf{e}}_4(\hat{\mathbf{q}}) \cdot \mathbf{g}_{\mathbf{q}}] + O\left(\frac{\omega_p}{\omega_B}\right) + O(q) \tag{5.5a}$$

$$A^+_{h\sigma,-\mathbf{q}} \simeq \frac{1}{\sqrt{2}} \frac{\beta}{\rho} [-\sigma \hat{\mathbf{e}}_3(-\hat{\mathbf{q}}) \cdot \mathbf{g}_{-\mathbf{q}} + i\hat{\mathbf{e}}_4(-\hat{\mathbf{q}}) \cdot \mathbf{g}_{-\mathbf{q}}] + O\left(\frac{\omega_p}{\omega_B}\right) + O(q) \tag{5.5b}$$

for the propagating plasma modes one has

$$A_{\nu\sigma,\mathbf{q}} = \frac{1}{\sqrt{2}} \left(\frac{\omega_p}{q} n_{\mathbf{q}} + \sigma \frac{|q_z|}{q_z} g_{z,q} + \frac{\alpha}{\rho \chi_T} \frac{q}{\omega_p} T_{\mathbf{q}}\right) + O\left(\frac{\omega_p}{\omega_B}\right) \tag{5.6a}$$

$$A^+_{\nu\sigma,-\mathbf{q}} = \frac{1}{\sqrt{2}} \frac{\beta}{\rho} \left(\frac{\omega_p}{q} n_{-\mathbf{q}} + \sigma \frac{|q_z|}{q_z} g_{z,-\mathbf{q}} \frac{\alpha\rho}{\chi_T} \frac{q}{\omega_p} T_{-\mathbf{q}}\right) + O\left(\frac{\omega_p}{\omega_B}\right) \tag{5.6b}$$

and for the heat mode one has

$$A_{H,\mathbf{q}} = -\frac{\gamma - 1}{\alpha} n_{\mathbf{q}} + O(q) \tag{5.7a}$$

$$A^+_{H,-\mathbf{q}} = \frac{\rho C_v}{k_{\mathrm{B}} T^2} T_{-\mathbf{q}} + O(q) \tag{5.7b}$$

The transport coefficients $\lambda(z)$ can be written as

$$\lambda(z) = \lambda^{\text{reg}}(z) + \delta\lambda(z) \tag{5.8a}$$

where $\lambda^{\text{reg}}(z)$ is the contribution to $\lambda(z)$ from the short- and intermediate-time regions, approximated here with $\lambda^0(z)$, and $\delta\lambda(z)$ is the contribution from the coupling of two hydrodynamic modes, given by

$$\delta\lambda(z) = \frac{1}{2} \sum_{\mu,\nu = h\sigma, \nu\sigma, H} \frac{1}{\Omega} \sum_{\mathbf{q}}' \lim_{\mathbf{k}\to 0} \frac{S_{\lambda\alpha}^{+\mu\nu}(\mathbf{q}, \mathbf{k}-\mathbf{q})\, S_{\lambda\beta}^{\mu\nu}(\mathbf{q}, \mathbf{k}-\mathbf{q})}{z + z_\mu(\mathbf{q}) + z_\nu(\mathbf{k}-\mathbf{q})} \tag{5.8b}$$

Similarly, the coefficients of self-diffusion $D_{\|}$ and $D_{\perp}$ can be written as

$$D_{\|(\perp)} = D_{\|(\perp)}^{\text{reg}} + \delta D_{\|(\perp)} \tag{5.9a}$$

where the hydrodynamic mode coupling contribution $\delta D_{\|(\perp)}$ is given by

$$\delta D_{\|(\perp)} = \sum_{\mu = h\sigma, \nu\sigma, H} \frac{1}{\Omega} \sum_{\mathbf{q}}' \frac{\langle v_{1z(x)} A_{\mu\mathbf{q}} C_{-\mathbf{q}} \rangle_{\text{eq}} \langle A_{\mu-\mathbf{q}}^{+} C_{\mathbf{q}} v_{1z(x)} \rangle_{\text{eq}}}{z_s(\mathbf{q}) + z_\mu(\mathbf{q})} \tag{5.9b}$$

where the $z(x)$ component of $\mathbf{v}_1$ refers to $\delta D_{\|}(\delta D_{\perp})$. In Eq. (5.9b) the limits $\mathbf{k} \to 0$ and $z \to 0$ have been taken.

In the remainder of this section we evaluate the two mode coupling contribution to the transport coefficients. Since the magnetic field increases the range of validity of hydrodynamics in the xy plane, we expect the two mode coupling contribution to the transverse transport coefficients to be important and, for strong enough fields, to dominate the bare or regular part. We are mainly interested here in the magnetic field dependence of the transport coefficients for strong fields. Consequently, we will neglect all mode coupling contributions that have the same magnetic field dependence as the corresponding regular parts and that simply represent higher order ε_p corrections to λ^{reg}.

The evaluation of the mode coupling contribution will be exemplified for $D_{\perp}$. Only the results will be given for all the other transport coefficients, since the calculation is analogous to that of $\delta D_{\perp}$. The relevant mode coupling amplitudes can be obtained from the results given in Ref. 21.

The amplitude for the coupling of the mode of self-diffusion with a heat mode vanishes. There are therefore only two mode coupling contributions to $D_{\|(\perp)}$, i.e., $\delta D_{\|(\perp)} = \delta D_{\|(\perp)}^{sh} + \delta D_{\|(\perp)}^{sv}$, where $\delta_{\|(\perp)}^{sh}$ and $\delta_{\|(\perp)}^{sv}$ arise from the coupling of the mode of self-diffusion with the upper hybrid modes and the propagating plasma modes, respectively. In general, the contribution to $\delta\lambda$ from the coupling of modes μ and ν will be denoted by $\delta\lambda^{\mu\nu}$.

The amplitudes are easily evaluated, with the result

$$\langle v_{1x} A_{h\sigma,\mathbf{q}} C_{-\mathbf{q}} \rangle_{\text{eq}} \langle A^{+}_{h\sigma,-\mathbf{q}} C_{\mathbf{q}} v_{1x} \rangle_{\text{eq}} = \frac{1}{2\rho\beta} + O\left(\frac{\omega_p}{\omega_B}\right)^2 \tag{5.10a}$$

and

$$\langle v_{1x} A_{v\sigma,\mathbf{q}} \rangle_{\text{eq}} \langle A^{+}_{v\sigma,-\mathbf{q}} C_{\mathbf{q}} v_{1x} \rangle_{\text{eq}} = \frac{1}{2\rho\beta} \frac{\omega_p^2}{\omega_B^2} \frac{q_y^2}{q^2} + O\left(\frac{\omega_p}{\omega_B}\right)^4 \tag{5.10b}$$

The corresponding mode coupling contributions are

$$\delta D_{\perp}^{sh} = \frac{1}{2\rho\beta} \sum_{\sigma=\pm 1} \int' \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{z_s(\mathbf{q}) + z_{h\sigma}(\mathbf{q})} \tag{5.11a}$$

$$\delta D_{\perp}^{vh} = \frac{1}{2\rho\beta} \frac{\omega_p^2}{\omega_B^2} \sum_{\sigma=\pm 1} \int' \frac{d\mathbf{q}}{(2\pi)^3} \frac{q_y^2}{q^2} \frac{1}{z_s(\mathbf{q}) + z_{v\sigma}(\mathbf{q})} \tag{5.11b}$$

where the bulk limit has been taken. The prime in Eqs. (5.11) denotes the cutoff in the $\mathbf{q}$ integration. The cutoffs for q_z and $q_\perp$ have to be chosen as the smallest among the cutoffs of the two hydrodynamic modes involved in the integral. The transport coefficients in the dispersion reltion on the right-hand side of Eqs. (5.11) are identified with their regular or bare part, λ^{reg}, approximated here with the Landau value, λ^0. Noth modes in Eq. (5.11a) have cutoffs $q_z^0 \sim l^{-1}$ and $q_\perp^0 \sim r_{\mathrm{L}}^{-1}$. Therefore we truncate the integration at $q_{zs}^0 = (v_c/D_{\|}^0)^{1/2}$ and $q_{\perp s}^0 = r_{\mathrm{L}}^{-1}$. Equation (5.11b) yields

$$\delta D_{\perp}^{sh} = \frac{1}{4\pi^2 \rho\beta} \frac{v_c}{r_{\mathrm{L}}^2 \omega_B^2} \left(\frac{v_c}{D_{\|}^0}\right)^{1/2} \left\{ \frac{D_{\|}^0}{2r_{\mathrm{L}}^2} \left[1 + \frac{\operatorname{Re} v_\perp^0(i\sigma\omega_B)}{D_\perp^0} \right] + \frac{1}{3} \left[1 + \frac{\operatorname{Re} v_{\|}^0(i\sigma\omega_B)}{D_{\|}^0} \right] \right\} + O\left(\frac{\omega_p}{\omega_B}\right)^2 \tag{5.12a}$$

Substituting the values of the bare transpot coefficients given in Eqs. (4.7) and (B.3), we find

$$\frac{\delta D_\perp^{sh}}{D_\perp^0} \simeq 0.6\varepsilon_p \frac{v_c}{\omega_p} \frac{\omega_B^2}{\omega_p^2} \tag{5.12b}$$

or $\delta D_\perp^{sh} \simeq O(\omega_B^0)$. We will discuss this result below.

The cutoffs for the mode coupling integral for $\delta D_\perp^{sv}$ have to be identified with those of the propagating plasma modes, for which $q_{zv}^0 \sim q_{\perp v}^0 \sim l^{-1}$. It follows that $\delta D_\perp^{sv} \sim O(\varepsilon_p^4 \, \omega_p^2/\omega_B^2)$. This represents a small plasma parameter correction to the bare diffusion coefficient $D_\perp^0$, and is therefore neglected.

The relevant amplitudes for the longitudinal diffusion coefficient $D_{\|}$ are

$$\langle v_{1z} A_{v\sigma,\mathbf{q}} C_{-\mathbf{q}} \rangle_{\text{eq}} \langle A^{+}_{v\sigma,-\mathbf{q}} C_{\mathbf{q}} v_{1z} \rangle_{\text{eq}} = \frac{1}{2\rho\beta} + \left(\frac{\omega_p}{\omega_B}\right)^2 \tag{5.13a}$$

$$\langle v_{1z} A_{h\sigma,\mathbf{q}} C_{-\mathbf{q}} \rangle_{\text{eq}} \langle A^{+}_{h\sigma,-\mathbf{q}} C_{\mathbf{q}} v_{1z} \rangle_{\text{eq}} = \frac{1}{2\rho\beta} \left(\frac{\omega_p}{\omega_B}\right)^4 + O\left(\frac{\omega_p}{\omega_B}\right)^6 \tag{5.13b}$$

and the corresponding mode coupling contributions are

$$\delta D_{\|}^{sv} = \frac{1}{2\rho\beta} \sum_{\sigma=\pm 1} \int' \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{z_s(\mathbf{q}) + z_v(\mathbf{q})} \sim O(\varepsilon_p^4 \omega_B^0) \tag{5.14a}$$

$$\delta D_{H}^{sh} = \frac{1}{2\rho\beta} \left(\frac{\omega_p}{\omega_B}\right)^4 \sum_{\sigma=\pm 1} \int' \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{z_s(\mathbf{q}) + z_{h\sigma}(\mathbf{q})} \sim O\left(\varepsilon_p^3 \frac{\omega_p^4}{\omega_B^4}\right) \tag{5.14b}$$

Since $D_{\|}^0 \sim O(\varepsilon_p^{-1} \omega_B^0)$, the above mode coupling contributions are either small-ε_p corrections to $D_{\|}^0$ or corrections in ω_p/ω_B. The relevant viscosities are the linear combinations appearing in the hydrodynamic modes, defined in Eqs. (3.5) and (3.6).

The only "interesting" mode coupling contribution to the viscosities for the upper hybrid modes is

$$\begin{aligned}
\delta v_{\perp}^{h\sigma h\sigma}(i\sigma\omega_B) &= \frac{1}{2\rho\beta} \sum_{\sigma',\sigma''=\pm 1} \int' \frac{d\mathbf{q}}{(2\pi)^3} \left[\frac{1}{2}(1+\sigma'\sigma'') + \frac{\sigma}{3}(\sigma'+\sigma'')\right. \\
&\quad \left. + \frac{1}{36}(1-\sigma'\sigma'')\right] \frac{1}{i\sigma\omega_B + z_{h\sigma'}(\mathbf{q}) + z_{h\sigma'}(\mathbf{q})} \\
&= \frac{1}{2\rho\beta} \int' \frac{d\mathbf{q}}{(2\pi)^3} \left[\frac{1}{q} \frac{1}{i\sigma\omega_B + 2\,\mathrm{Re}\, z_{h+}(\mathbf{q})}\right. \\
&\quad \left. + \sum_{\sigma'=\pm 1} \left(1 + \frac{2}{3}\sigma\sigma'\right) \frac{1}{i\sigma\omega_B + 2z_{h\sigma'}(\mathbf{q})}\right]
\end{aligned} \tag{5.15a}$$

All the other mode coupling contributions to both $v_{\perp}$ and $v_{\|}$ do not lead to any new magnetic field dependence. The right-hand side of Eq. (5.15a) is easily evaluated, with the result

$$\mathrm{Re}\, \delta v_{\perp}^{h\sigma h\sigma}(i\sigma\omega_B) \simeq \frac{19}{36\pi^2 n} \left(\frac{v_c}{v_{\|}^0}\right)^{1/2} \left[\frac{v_c}{3} + \frac{1}{2r_{\mathrm{L}}^2} \mathrm{Re}\, v_{\perp}^0(i\sigma\omega_B)\right] \tag{5.15b}$$

$$\mathrm{Im}\, \delta v_{\perp}^{h\sigma h\sigma}(i\sigma\omega_B) = -\frac{\sigma}{72\pi^2 n} \left(\frac{v_c}{v_{\|}^0}\right)^{1/2} \left[31\omega_B + \frac{19\sigma}{r_{\mathrm{L}}^2} \mathrm{Im}\, v_{\perp}^0(i\sigma\omega_B)\right] \tag{5.15c}$$

where use has been made of the fact that $v_{\|}^{0} = v_{\|}^{0}(i\sigma\omega_B)$ is real. Using Eqs. (3.5b) and (B.1), one then obtains

$$\frac{\operatorname{Re}\,\delta v_{\perp}^{h\sigma h\sigma}(i\sigma\omega_B)}{\operatorname{Re}\,v_{\perp}^{0}(i\sigma\omega_B)} \simeq 0.56\varepsilon_p \frac{v_c}{\omega_p}\frac{\omega_B^2}{\omega_p^2} \tag{5.15d}$$

$$\frac{\operatorname{Im}\,\delta v_{\perp}^{h\sigma h\sigma}(i\sigma\omega_B)}{\operatorname{Im}\,v_{\perp}^{0}(i\sigma\omega_B)} \simeq 0.76\varepsilon_p \frac{v_c}{\omega_p}\frac{\omega_B^2}{\omega_p^2} \tag{5.15e}$$

We have then $\operatorname{Re}\,\delta v_{\perp}^{h\sigma h\sigma} \sim O(\omega_B^0)$ and $\operatorname{Im}\,\delta v_{\perp}^{h\sigma h\sigma} \sim O(\omega_B)$. The mechanism responsible for his anomalous contribution to the transverse viscosity is analogous to that determining the mode coupling contribution to $D_{\perp}$. There is no anomalous mode coupling contribution to $v_{\|}(i\sigma\omega_B)$, nor to the viscosities for the propagating plasma modes.

Finally, we consider the coefficients of thermal conductivity $D_{\|}^{T}$ and $D_{\perp}^{T}$. Once more, the relevant role is played by the upper hybrid modes that lead to the following anomalous mode coupling contribution to $D_{\perp}^{T}$:

$$\delta D_{\perp}^{T\,h\sigma H} = \frac{1}{\rho\beta}\left[1 + \left(\frac{\partial p}{\partial u}\right)_{\rho}\right]^2 \sum_{\sigma=\pm 1} \int' \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{z_H(\mathbf{q}) + z_{h\sigma}(\mathbf{q})} \tag{5.16a}$$

For consistency we use the ideal gas values in Eq. (16a), $(\partial p/\partial u)_{\rho} = 2/3$. Again, the cutoffs of upper hybrid and heat modes are of the same order. We obtain

$$\delta D_{\perp}^{T\,h\sigma H} \simeq \frac{25}{9\pi^2 n}\left(\frac{v_c}{v_{\|}^{0}}\right)^{1/2} \left\{\frac{v_c}{6}\left[1 + \frac{D_{\|}^{T0}}{v_{\|}^{0}(i\sigma\omega_B)}\right] + \frac{1}{4r_{\mathrm{L}}^2}\left[\operatorname{Re}\,v_{\perp}(i\sigma\omega_B) + D_{\perp}^{T0}\right]\right\} \tag{5.16b}$$

or

$$\frac{\delta D_{\perp}^{T\,h\sigma H}}{D_{\perp}^{T0}} \simeq 0.16\varepsilon_p \frac{v_c}{\omega_p}\frac{\omega_B^2}{\omega_p^2} \tag{5.16c}$$

Again we have $\delta D_{\perp}^{T\,h\sigma H} \simeq O(\omega_B^0)$.

We conclude this section with a few comments on the results obtained. In contrast with the previous literature,[3–6] we find that it is always a coupling mechanism involving the upper hybrid modes that gives rise to an anomalous contribution, of $O(\omega_B^0)$, to the transport coefficients $D_{\perp}$, $D_{\perp}^{T}$, and $\operatorname{Re}\,v_{\perp}(i\sigma\omega_B)$. This is because the upper hybrid modes can be extended to $q_{\perp} \sim r_{\mathrm{L}}^{-1} = m\beta\omega_B$, while the cutoff for the propagating plasma modes in the xy plane is $\sim l^{-1}$. In the literature it was always assumed *a priori* that high-frequence modes such as the hybrid modes are not responsible for an enhancement of the transport coefficients[3,4] via a mode coupling mechanism. There is, however, no reason for believing that the Kadanoff–Swift arguments break down for finite-frequency modes that are hydrodynamic in character, since their lifetime diverges as $k \to 0$. It has also been shown elsewhere that a fluctuating hydrodynamics approach leads to results that are identical to those given here.[25]

Physically the dominant role of the upper hybrid modes can be understood as follows. For large field these modes are similar to shear waves in neutral fluids, since, to leading order in ω_p/ω_B, they represent velocity fluctuations in the xy plane. More precisely, they are associated with vortices in the plane. Such vortices are excited over a wide range of length scale, down to r_L, and are very effective in enhancing the transport properties.

Finally, we notice that for very strong fields, such that $\omega_p/\omega_B \simeq [\varepsilon_p^2 \ln(\varepsilon_p^{-1})]^{1/2}$, the right-hand side of Eqs. (5.12b), (5.15d), and (5.15e), and (5.16c) is of $O(1)$, i.e., the mode coupling corrections become of the same order as the bare transport coefficients. For larger magnetic fields the two-mode-coupling approximation is no longer adequate and more complicated mode coupling effects need to be taken into account. This will be done in an approximate way in the next section by employing a "self-consistent" mode coupling theory.

6. SELF-CONSISTENT MODE COUPLING THEORY

An approximate way of taking into account the coupling of more than two hydrodynamic modes is to replace all the transport coefficients appearing on both the left- and the right-hand sides of the mode coupling equations by their mode coupling values and solve simultaneously the entire set of equations.[17] It is important to note that when the two-mode coupling theory becomes inadequate, our estimate of the hydrodynamic cutoffs also breaks down. The latter were in fact evaluated using bare transport coefficients. In the very strong-field regime of interest here the transverse transport coefficients differ considerably from their bare value. Consequently, in a self-consistent mode coupling theory the cutoffs must also be expressed in terms of the mode coupling value of the transport coefficients. Denoting by $\delta\lambda^{sc}$ the self-consistent mode coupling correction to the transport coefficient λ, with $\lambda = \lambda^0 + \delta\lambda^{sc}$ in this regime, the set of mode coupling equations is

$$\delta D_\perp^{sc} \simeq \frac{1}{4\pi^2 \rho\beta} \operatorname{Re} \int_0^{(v_c/D_\parallel^0)^{1/2}} \int_0^{(v_c/D_\perp^{sc})^{1/2}} dq_\perp\, q_\perp \{ i\omega_B + (v_\parallel^0 + D_\parallel^0) q_z^2 + [\delta D_\perp^{sc} + \delta v_\perp^{sc}(i\omega_B)] q_\perp^2 \}^{-1} \tag{6.1a}$$

$$\delta v_\perp^{sc}(i\sigma\omega_B) \simeq \frac{1}{4\pi^2\rho\beta}\int_0^{(v_c/v_\parallel^0)^{1/2}}\int_0^{([v_c/\mathrm{Re}\,\delta v_\perp^{sc}(i\omega_B)]^{1/2}} dq_\perp\, q_\perp \left\{\frac{1}{9}\,[i\sigma\omega_B + 2v_\parallel^0 q_z^2 + 2\,\mathrm{Re}\,\delta v_\perp^{sc}(i\omega_B)\, q_\perp^2]^{-1} + \sum_{\sigma'=\pm 1}\left(1+\frac{2}{3}\sigma'\right)[i(\sigma+2\sigma')\omega_B + 2v_\parallel^0 q_z^2 + 2\delta v_\perp^{sc}(i\sigma'\omega_B)]^{-1}\right\} \tag{6.1b}$$

$$\delta D_\perp^{T,sc} \simeq \frac{1}{\pi^2\rho\beta}\left[1+\left(\frac{\partial p}{\partial u}\right)_\rho\right]\mathrm{Re}\int_0^{(v_c/D_\parallel^{T0})^{1/2}}\int_0^{(v_c/\delta D_\perp^{T,sc})^{1/2}} dq_\perp\, dq_\perp \times [i\omega_B + (v_\parallel^0 + D_\perp^{T,0})q_z^2 + [\delta v_\perp^{sc}(i\omega_B) + \delta D_\perp^{T,sc})q_\perp^2]^{-1} \tag{6.1c}$$

In Eqs. (6.1) the longitudinal transport coefficients are still approximated by their bare Landau values, since their mode coupling corrections are much smaller than their bare values for the magnetic field used here. The set of equations (6.1) is easily solved, with the result that $\delta D_\perp^{sc}$, $\mathrm{Re}\,\delta v_\perp^{sc}(i\omega_B)$, and $\delta D_\perp^{T,sc} \sim B^{-1}$. More precisely, one finds

$$\delta D_\perp^{sc} \simeq \alpha_D \frac{k_B T}{m\omega_B}\,\varepsilon_p^{1/2}(\varepsilon_p \ln \varepsilon_p^{-1})^{3/2} \tag{6.2a}$$

$$\mathrm{Re}\,\delta v_\perp^{sc}(i\omega_B) \simeq \alpha_v \frac{k_B T}{m\omega_B}\,\varepsilon_p^{1/2}(\varepsilon_p \ln \varepsilon_p^{-1})^{3/2} \tag{6.2b}$$

$$\delta D_\perp^{T,sc} \simeq \alpha_T \frac{k_B T}{m\omega_B}\,\varepsilon_p^{1/2}(\varepsilon_p \ln \varepsilon_p^{-1})^{3/2} \tag{6.2c}$$

where α_D, α_v, and α_T are numerical constants, given by $\alpha_D \simeq 0.5$; $\alpha_v \simeq 3.1$, and $\alpha_T \simeq 1.5$. For large enough magnetic field the self-consistent mode coupling contribution dominates all other contribution and all the above transverse coefficients are observed to have a Bohm-like $\sim\omega_B^{-1}$ behavior. An estimate of the value of B where the Bohm-like behavior dominates is given by the value of B, where the bare hydrodynamic cutoffs have to be replace with their mode coupling values, and is found to be $\omega_p/\omega_B \sim 0.25(\varepsilon_p^2 \ln \varepsilon_p^{-1})^{1/2}$.

A a result of the analysis presented here, the magnetic field dependence of the transverse transport coefficients $D_\perp$, $D_\perp^T$, and $\mathrm{Re}\, v_\perp(i\omega_B)$, collectively denoted by $\lambda_\perp$, can be described as follows:

(a) A classical region, described by the BGL kinetic theory, where $\lambda_\perp \sim B^{-2}$, for

$$2.5(\varepsilon_p^2 \ln \varepsilon_p^{-1})^{1/2} < \omega_p/\omega_B < \nu_c/\omega_B$$

(b) A plateau region, where the two hydrodynamic mode coupling contribution to the transverse transport coefficients dominates and $\lambda_\perp \sim B^0$,

$$(\varepsilon_p^2 \ln \varepsilon_p^{-1})]^{1/2} < \omega_p/\omega_B < 2.5(\varepsilon_p^2 \ln \varepsilon_p^{-1})^{1/2}$$

The mode coupling mechanism responsible for the enhancement of $\lambda_\perp$ is always one that involves the finite-frequency upper hybrid modes.

(c) A Bohm-like region, where a self-consistent mode coupling theory is needed and $\lambda_\perp \sim B^{-1}$, for

$$\omega_p/\omega_B < 0.25(\varepsilon_p^2 \ln \varepsilon_p^{-1})^{1/2}$$

7. DISCUSSION

We have evaluated the hydrodynamic mode coupling contribution to the transport coefficients of a three-dimensional OCP in an external magnetic field. We have shown that the long-lived collective excitations of the system are responsible for an enhancement of the transport rates in the plane orthogonal to the field over their BGL values. The transverse coefficients of self-diffusion $D_\perp$, heat conductivity $D_\perp^T$, and kinemtic viscosity $\operatorname{Re} \nu_\perp(i\omega_B)$ were found to behave as B^{-1} in the strong-field regime. This behavior is observed experimentally in a variety of systems. The relevant mode coupling mechanism is always one that involves the finite-frequency upper hybrid modes. This mechanism was neglected in the previous literature, where, as a consequence, no Bohm-like behavior was predicted for the transport coefficients of an infinite three-dimensional electron gas.[3-6]

The details of the results obtained here have been summarized at the end of the previous section. It is important to stress that the quantitative predictions of the mode coupling theory are strongly dependent on the values of the hydrodynamic cutoffs. We have obtained a reliable estimate of the magnetic field and plasma parameter dependence of the cutoffs, but we do not know their exact value. Consequently, the numerical coefficients in Eqs. (6.2) are not precisely known. The exact coefficients can in principle be obtained by using kinetic theory. We stress, however, that the magnetic field dependence of the mode coupling contributions does not depend on the introduction of cutoffs, nor on the small plasma approximation used in their evaluation.

Several authors[2-7] have studied the problem of transport in plasmas in a strong external magnetic field, with attention to the transverse coefficient of self-diffusion $D_\perp$. Krommes and Oberman[6] have employed a self-consistent mode coupling theory to study the two-dimensional problem. In two dimensions only the upper hybrid modes are propagating modes. The two propagating plasma modes are replaced by a single diffusive mode, usually referred to as the "convective cells mode." Such a mode can be extended to $k_\perp \sim r_L^{-1}$. The dominant mode coupling effect is then due to the coupling of the convective cell mode with the mode of self-diffusion and leads to three well-defined regions ($\sim B^{-2}$, B^0, B^{-1}) in the mgnetic field dependence of $D_\perp$. The contribution from the coupling of the upper hybrid modes with the diffusive mode is negligible in this case for small ε_p because it is of higher order in the plasma parameter. The results of Ref. 6 are in good agreement with the two-dimensional numerical simulation of Okuda and Dawson.[4] In contrast, we find that in three-dimensions (3d) the dominant effect is due to the upper hybrid modes.

The 3d problem has also been considered by Okuda and Dawson[4,5] and Montgomery *et al.*[3] All these authors effectively reduced the problem to a 2d one by considering the case where the size of the system in the direction of the field L_z is small. They evaluated a contribution to $\delta D_\perp$, denoted here by $\delta D_\perp^{(L)}$, due to the coupling of the propagation of plasma waves with the diffusion modes that behaves as B^{-1} for very strong fields, but is size-dependent and vanishes in the thermodynamic limit ($L_z \to \infty$). As discussed in Ref. 1, $\delta D^{sh\sigma}$ evaluated here dominates $\delta D_\perp^{(L)}$ only if $L_z/l \geqslant \varepsilon_p^{-2}$. This implies that our calculation is not relevant for laboratory plasma, where ε_p is very small. Furthermore, in all 3D computer experiments of Okuda and Dawson L_z is only a few ean free paths and the measured effect is indeed the finite-size contribution, as discussed by the authors. We will return to the question of the observability of our result below.

Recently Rose[7] has used a fluctuating hydrodynamic approach to compute $D_\perp$ for 3d systems. He also finds that the contribution from hydrodynamic fluctuations leads to a correction to the BGL value of $D_\perp$ that is independent of the magnetic field. He does not however discuss the need for a self-consistent theory at stronger fields.

The anomalous high-B contribution to the transverse transport rates evaluated here should be observable in a solid -state plasma, where the density of electron carriers at room temperature can be $\sim 6 \times 10^{13}\,\text{cm}^{-3}$ and the plasma parameter can then be large enough for our bulk effect of dominate the finite-size one. Furthermore, the two effects have a different ε_p dependence: their ratio is of order ε_p^{-2}. Consequently, it might be possible to see the transition from essentially 2d to bulk 3d behavior by changing the size or the parameters of the system. The 3d behavior should be observable for $L_z > 10^{-2}$ cm for a plasma at room temperature. Possible candidates include solid-state plasmas, but they are multicomponent systems and can sustain hydrodynamic modes not present in the OCP. It is possible that a new mode coupling mechanism may lead to an extra contribution to the transverse transport coefficients. To answer this, one needs to consider the mode coupling theory for two- or multicomponent systems.

APPENDIX A

In this Appendix we list eigenfunctions and eigenvalues of the hydrodynamic matrix $\mathscr{L}_{ij}$ for arbitrary values of the parameter ω_p/ω_B. The form of right and left eigenfunctions for $\omega_p/\omega_B \ll 1$ is also given.

The eigenvalues are the solutions of the determinant (3.2d). Eigenfunctions and eigenvalues are assumed to have a power series expansion in k of the form

$$z_\mu(\mathbf{k}) = z_\mu^{(0)} + z_\mu^{(1)} + z_\mu^{(2)} + O(k^3) \tag{A.1a}$$

and

$$\theta_\mu^{\mathrm{R,L}}(\mathbf{k}) = \theta_\mu^{\mathrm{R,L}(-1)} + \theta_\mu^{\mathrm{R,L}(0)} + \theta_\mu^{\mathrm{R,L}(1)} + O(k^2) \tag{A.1b}$$

where $z_\mu^{(n)}$ and $\theta_\mu^{\mathrm{R,L}(n)}$ are of order k^n. Due to the singular small-k behavior of the direct correlation function [responsible for the terms of $O(k^{-1})$ in Eqs. (2.20b) and (2.20c)], terms of $O(k^{-1})$ appear in the eigenvalues. There are five modes: four finite-frequency modes (i.e., modes whose propagating part does not vanish with k) and one diffusive mode. For the two upper hybrid modes, labeled with $h\sigma$, with $\sigma = \pm 1$, and the two propagating plasma modes, labeled with $\nu\sigma$, with $\sigma = \pm 1$, we find $z_\mu^{(2n+1)} = 0$, and

$$z_{h\sigma}^{(0)}(\mathbf{k}) = \frac{i\sigma}{\sqrt{2}}\,\omega_h \left[1 + \left(1 - 4\hat{k}_z^2\,\frac{\omega_p^2\omega_B^2}{\omega_h^4}\right)^{1/2}\right]^{1/2} \tag{A.2}$$

$$z_{\nu\sigma}^{(0)}(\mathbf{k}) = \frac{i\sigma}{\sqrt{2}}\,\omega_h \left[1 - \left(1 - \hat{k}_z^2\,\frac{\omega_p^2\omega_B^2}{\omega_h^4}\right)^{1/2}\right]^{1/2} \tag{A.3}$$

Here $\omega_h = (\omega_p^2 + \omega_B^2)^{1/2}$ is known as the upper hybrid frequency. The terms of $O(k^2)$ are given in terms of the terms of $O(k^0)$ as follows:

$$z_\mu^{(2)}(\mathbf{k}) = \frac{z_\mu^{(0)}}{2[2(z_\mu^{(0)})^2 + \omega_h^2]} \left\{ -\frac{\gamma}{\rho\chi_T} k^2 \left[1 + \frac{\omega_B^2}{(z_\mu^{(0)})^2} \hat{k}_z^2\right] + 2\omega_B \left(1 + \frac{\omega_p^2}{(z_\mu^{(0)})^2} \hat{k}_z^2\right) (v_{3\mu} k_\perp^2 + v_{4\mu} k_z^2)\right\}$$
$$+ \frac{1}{2[2(z_\mu^{(0)})^2 + \omega_h^2]} \left\{ z_\mu^{(0)2} \left[v_{2\mu} k^2 + k_\perp^2 \left(v_{1\mu} + \frac{v_{0\mu}}{3}\right) + \frac{4}{3} v_{0\mu} k_z^2 \right] + \omega_p^2 \hat{k}_z^2 k_\perp^2 \left(v_{1\mu} + 3v_{0\mu} - \frac{2\omega_B}{z_\mu^{(0)}} v_{4\mu} - 2v_{2\mu}\right) - \omega_B^2 \left(1 + \frac{\omega_p^2}{(z_\mu^{(0)})^2} \hat{k}_z^2\right) (v_{1\mu} k_\perp^2 + v_{2\mu} k_z^2)\right\} \quad \text{(A.4a)}$$

All the kinematic viscosities in Eq. (A.4a) are finite-frequency viscosities, evaluated at $z_\mu^{(0)}$, i.e.,

$$v_{j\mu} \equiv v_j(z_\mu^{(0)}) \quad \text{(A.4b)}$$

for $j = 0, 1, ..., 4$.

The left and right eigenvalues of the four modes above are, respectively,

$$\langle \theta_\mu^{\mathrm{L}}(\mathbf{k})| \simeq C_\mu(\hat{\mathbf{k}}) \left(\frac{i\omega_p^2}{z_\mu^{(0)} k_\perp} \left[1 + \hat{k}_z^2 \frac{\omega_B^2}{(z_\mu^{(0)})^2}\right], \frac{k_z}{k_\perp}\left[1 + \frac{\omega_B^2}{(z_\mu^{(0)})^2}\right], 1, -\frac{\omega_B}{z_\mu^{(0)}}, \frac{\alpha}{\rho\chi_T} \frac{ik^2}{z_\mu^{(0)} k_\perp} \left[1 + \hat{k}_z^2 \frac{\omega_B^2}{(z_\mu^{(0)})^2}\right]\right) \quad \text{(A5.a)}$$

and

$$|\theta_\mu^{\mathrm{R}}(\mathbf{k})\rangle = C_\mu(\hat{\mathbf{k}}) \begin{bmatrix} ik_\perp z_\mu^{(0)}/[(z_\mu^{(0)})^2 + \omega_p^2 \hat{k}_z^2] \\ -\omega_p^2 \hat{k}_z^2 \hat{k}_\perp/[(z_\mu^{(0)})^2 + \omega_p^2 \hat{k}_z^2] \\ 1 \\ \omega_B/z_\mu^{(0)} \\ [(\gamma - 1)/\alpha]\, ik_\perp z_\mu^{(0)}/[(z_\mu^{(0)})^2 + \omega_p^2 \hat{k}_z^2] \end{bmatrix} \quad \text{(A.5b)}$$

where $C_\mu(\hat{\mathbf{k}})$ is a normalization constant, given by

$$C_\mu(\hat{\mathbf{k}}) = \frac{1}{\sqrt{2}} z_\mu^{(0)} \frac{(z_\mu^{(0)})^2 + \omega_p^2 \hat{k}_z^2}{(z_\mu^{(0)})^2 \omega_h^2 + 2\omega_p^2 \omega_B^2 \hat{k}_z^2} \quad \text{(A.5c)}$$

The left and right eigenfunctions are given here in the form of row and column vectors, respectively. They are normalized according to

$$\langle \theta_\mu^{\mathrm{L}}(\mathbf{k}) | \theta_\nu^{\mathrm{R}}(\mathbf{k}) \rangle \varepsilon \delta_{\mu\nu} + O(k_2) \quad \text{(A.6a)}$$

with

$$\langle a|b\rangle = \sum_{j=1}^{5} a_j b_j \tag{A.6b}$$

There is a single diffusive heat mode with dispersion relation

$$z_H(\mathbf{k}) = D_{\parallel}^T k_z^2 + D_{\perp}^T k_{\perp}^2 + O(k^4) \tag{A.7a}$$

and left and right eigenfunctions given by

$$\langle \theta_H^{\mathrm{L}}(\mathbf{k})| = \left(-\frac{\gamma-1}{\alpha}, 0, 0, 0, 1 \right) + O(k) \tag{A.7b}$$

and

$$|\theta_H^{\mathrm{R}}(\mathbf{k})\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + O(k) \tag{A.7c}$$

The only modification introduced in the heat mode by the presence of the magnetic field is the anisotropy.

APPENDIX B

In this Appendix we list the transport coefficients appearing in Eqs. (3.4) in the limit of small plasma parameter, as obtained from the Landau form of the Balescu–Guernsey–Lenard kinetic equation.

The Landau approximation for the zero-frequency transport coefficients has been evaluated before.[10] It is, however, essential here to consider the case of finite-frequency transport coefficients, since they have a different magnetic field and plasma parameter dependence than their zero-frequency form. The viscosities for the upper hybrid modes are

$$\nu_0^0(i\sigma\omega_B) \simeq \frac{i\sigma}{m\beta\omega_B} + \frac{1}{m\beta\omega_B^2}\frac{2\nu_c}{5\sqrt{\pi}} \tag{B.1a}$$

$$\nu_1^0(i\sigma\omega_B) \simeq \frac{i\sigma}{3m\beta\omega_B} + \frac{1}{m\beta\omega_B^2}\frac{2\nu_c}{9\sqrt{\pi}} \tag{B.1b}$$

$$\nu_2^0(i\sigma\omega_B) \simeq \frac{i\sigma}{4m\beta\omega_B} + \frac{1}{m\beta\nu_c}\frac{5\sqrt{\pi}}{4} \tag{B.1c}$$

$$\nu_3^0(i\sigma\omega_B) \simeq \frac{2}{3m\beta\omega_B} + \frac{i\sigma}{m\beta\omega_B^2}\frac{8\nu_c}{45\sqrt{\pi}} \tag{B.1d}$$

$$\nu_4^0(i\sigma\omega_B) \simeq \frac{1}{4m\beta\omega_B} + \frac{i\sigma}{m\beta\nu_c}\frac{5\sqrt{\pi}}{4} \tag{B.1e}$$

$$\nu_{\|}^0(i\sigma\omega_B) \simeq 2\,\mathrm{Re}\,\nu_2^0(i\sigma\omega_B) \simeq \frac{1}{m\beta\nu_c}\frac{5\sqrt{\pi}}{2} \tag{B.2a}$$

$$\nu_{\perp}^0(i\sigma\omega_B) \simeq \frac{7\nu_c}{m\beta\omega_B^2\, 15\sqrt{\pi}} - \frac{5i\sigma}{6m\beta\omega_B} \tag{B.2b}$$

In all of the above the leading nonvanishing terms in ω_p/ω_B have been kept. The viscosities for the propagating plasma modes are

$$\nu_0^0(i\sigma\omega_p|\hat{k}_z|) = \frac{1}{m\beta}\left(\frac{2\nu_c}{5\sqrt{\pi}} - i\sigma\omega_p|\hat{k}_z|\right)^{-1} \tag{B.3a}$$

$$\nu_1^0(i\sigma\omega_p|\hat{k}_z|) \simeq \frac{1}{m\beta\omega_B^2}\frac{\nu_c}{10\sqrt{\pi}} - \frac{i\sigma\omega_p|\hat{k}_z|}{4m\beta\omega_B^2} \tag{B.3b}$$

$$\nu_2^0(i\sigma\omega_p|\hat{k}_z|) \simeq \frac{1}{m\beta\omega_B^2}\frac{2\nu_c}{5\sqrt{\pi}} - \frac{i\sigma\omega_p|\hat{k}_z|}{m\beta\omega_B^2} \tag{B.3c}$$

$$\nu_3^0(i\sigma\omega_p|\hat{k}_z|) \simeq \frac{1}{2m\beta\omega_B} \tag{B.3d}$$

$$\nu_4^0(i\sigma\omega_p|\hat{k}_z|) \simeq \frac{1}{m\beta\omega_B} \tag{B.3e}$$

Equation (B.3a) is exact as a function of ω_B. The imaginary parts of ν_3 and ν_4 are neglected because they are of order $(\omega_p/\omega_B)^3$. For completeness, the two coefficients of thermal conductivity are

$$D_{\|}^{T0} \simeq \frac{1}{m\beta\nu_c}\frac{10\sqrt{\pi}}{3} \tag{B.4a}$$

$$D_{\perp}^{T0} \simeq \frac{1}{m\beta\omega_B^2}\frac{5\nu_c}{6\sqrt{\pi}} \tag{B.4b}$$

The coefficients of self-diffusion have been given in Eqs. (4.6).

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